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A STELLAR MODEL OF MIXED OPACITY AND ITS VARIATIONS WITH THE MASS, CHEMICAL COMPOSITION, OPACITY COEFFICIENTS, AND ENERGY-GENERATION COEFFICIENT

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SUMMARY

Kushwaha's stellar model of 10 solar masses has been reinvestigated in such a way that the luminosity and the radius are determined simultaneously by integrating the differential equations of the stellar interior. The effect on the luminosity and the radius of the model when small changes in the mass, chemical composition, opacity coefficients, and energy-generation coefficient are applied has been examined by varying the parameters which enter into these differential equations. A mass-luminosity and a mass-radius relation have been derived for the models near ten solar masses.

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INTRODUCTION

Stellar models of mixed opacity (that due to both electron scattering and the modified Kramers' law, combined by straight addition), with radiation pressure taken into account, have been integrated by Härm and Schwarzschild (Reference 1), studied in detail by Kushwaha (Reference 2), and later studied by Morton (Reference 3) and by Savedoff and van Dyck (Reference 4). As is customary and justified for the main sequency stars of fairly large masses, those authors have assumed that the generation of thermonuclear energy is confined to the convective core. In this way, Kushwaha has studied not only the homogeneous model of the zero-age main sequence star but also the inhomogeneous models in the early stage of evolution after the main sequence.

In this report no attempt will be made to study the evolution, but rather the calculation of the homogeneous model will be put on a numerically more accurate base than has been done before and an attempt will be made to understand the effect of small changes in the mass, chemical composition, opacity coefficients, and energy-generation coefficient on the nature of the star.

EQUATIONS OF THE ENVELOPE

Following the notation of Schwarzschild (Reference 5) we may write the equations of mixed opacity arising from the atomic absorption given by Kramers' law and from electron scattering as

$$\kappa = \kappa_1\left(\frac{\rho}{T^{3.5}}\right) + \kappa_2 , \tag{1}$$

$$\left.\begin{aligned} \kappa_1 &= \kappa_1' Z(1 + X) \ , \\ \kappa_2 &= \kappa_2'(1 + X) \ , \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} \kappa_1' &= 3.08 \times 10^{25}\, d \ , \\ \kappa_2' &= 0.19\, d \ , \end{aligned}\right\} \quad (3)$$

and the corresponding equations for the radiative envelope as

$$\left.\begin{aligned} \frac{dp}{dx} &= -\frac{\beta pq}{x^2 t} \ , \\ \frac{dq}{dx} &= \frac{x^2 \beta p}{t} \ , \end{aligned}\right\} \quad (4)$$

$$\frac{dt}{dx} = -C\left(\frac{\beta^2 p^2}{x^2 t^{8.5}} + F\frac{\beta p}{x^2 t^4}\right) , \quad (5)$$

$$\beta = 1 - B\left(\frac{t^4}{p}\right) , \quad (6)$$

$$B = \frac{4\pi a}{3}\left(\frac{H}{k}\right)^4 G^3 \mu^4 M^2 \ , \quad (7)$$

$$C = \frac{3\kappa_1}{4ac}\, \frac{1}{(4\pi)^3}\left(\frac{k}{GH}\right)^{7.5} \frac{LR^{0.5}}{\mu^{7.5} M^{5.5}} \ , \quad (8)$$

$$F = \frac{\kappa_2}{\kappa_1}(4\pi)\left(\frac{HG}{k}\right)^{3.5} \frac{M^{2.5}\mu^{3.5}}{R^{0.5}} \ . \quad (9)$$

The meanings of different symbols in these equations as well as the numerical values κ_1' and κ_2' follow those given in Kushwaha's paper (Reference 2). The constant factor d introduced by him assumes a value of 1.21 and was obtained by him by comparing the computed values of the opacity from his model with those from the opacity tables by Keller and Meyerott (Reference 6). It will be noted also that F as defined here is equivalent to the product AB in Kushwaha's paper. For the integration of the envelope, the constants B and F may be regarded as given and C may be treated as an eigenvalue that has to be found by integration in the usual way of fitting the radiative envelope on the convective core.

BOUNDARY CONDITIONS

Kushwaha has integrated these equations by imposing the boundary conditions

$$\left.\begin{aligned} p &= 0 , \\ q &= 1 , \\ t &= 0 , \\ \beta &= 0 , \end{aligned}\right\} \qquad (10)$$

at $x = 1$, and a proper fit at the interface between the core and the radiative envelope. It is obvious from the nature of the differential equations and the boundary conditions given by Equations 10 that in order to start the integration from the surface inward it is necessary to derive an expansion near $x = 1$. However, the boundary expansion derived in the usual way does not meet the practical purpose because of the smallness of the eigenvalue C. Kushwaha apparently used the starting formulas given by Härm and Schwarzschild, which are derived just for starting integration but are not rigorous. As a minor variation from Kushwaha's procedure, we will use the photospheric surface as the boundary of the star. Consequently we have the following boundary conditions at $x = 1$:

$$\left.\begin{aligned} p &= p_0 , \\ q &= 1 , \\ t &= t_0 , \end{aligned}\right\} \qquad (11)$$

where p_0 and t_0 denote respectively the pressure and the temperature on the surface of the photosphere. Moreover, we will take t_0 to be equal to the effective temperature of the star.

It should be pointed out that t_0 may not be arbitrarily chosen in the beginning of integration because it is related to the total luminosity and the radius of the star. In other words, t_0 is determined not only by the constants B and F but also by the eigenvalue C.

Since the luminosity L is given by

$$L = 4\pi R^2 \sigma T^4 , \qquad (12)$$

where T is the effective temperature in degrees Kelvin, we obtain from Equations 8 and 12, by eliminating L and converting T into the Schwarzschild variable t_0,

$$t_0^4 = \frac{16}{3\kappa_1} (4\pi)^2 \left(\frac{HG}{k}\right)^{3.5} \mu^{3.5} M^{1.5} R^{1.5} C . \qquad (13)$$

In what follows we will deal directly with the parameters B and F instead of M and R because only the former enter Equations 4 through 6. Eliminating M and R from Equations 7, 9 and 13, we obtain the boundary temperature in terms of B, F, and C. Thus,

$$t_0^4 = \frac{32(3)^{\frac{7}{2}}}{a^{\frac{9}{2}}} \pi^{\frac{1}{2}} \frac{\kappa_2^3}{\kappa_1^4} \left(\frac{k}{H}\right)^4 G^{\frac{1}{2}} \frac{B^{\frac{9}{2}} C}{\mu^4 F^3} . \qquad (14)$$

It is well known that the boundary conditions do not affect seriously the resulting configuration. Nevertheless, for a reason which will be discussed later, the exact effect of the boundary conditions on the structure of the model is examined first. From Kushwaha's paper, F and B are taken to be $F = 1.12204 \times 10^3$ and $B = 6.3096$, and the structure of several models is derived by assigning different values of β_0 (or equivalently p_0) at the boundary $x = 1$. Since t_0 is given by Equation 14, the boundary conditions are completely determined by β if the chemical composition of the model is specified; Kushwaha's values are used:

$$\left.\begin{aligned} X &= 0.90 , \\ Y &= 0.09 , \\ Z &= 0.01 . \end{aligned}\right\} \qquad (15)$$

The numerical integration of the equations was carried out on the IBM 704 digital computer at the National Bureau of Standards. The usual fitting procedure at the interface was carried out by the machine. The table representing the configuration of the polytropic gas sphere of index 1.5 was stored in the machine. Hence, the invariants derived from the computed values,

$$\left.\begin{aligned} v &= \frac{2}{5} \frac{\beta q}{x t} , \\ w &= \frac{2}{5} \frac{x^2 \beta^2 p}{t^2} , \end{aligned}\right\} \qquad (16)$$

and the same invariants from the tabulated values of Emden variables (for example, Reference 7),

$$\left.\begin{aligned} v &= -\frac{\xi\theta'}{\theta}\,, \\ w &= \xi^2\theta^{\frac{1}{2}}\,, \end{aligned}\right\} \qquad (17)$$

can be compared by means of interpolation. The computed envelope was regarded as fitted to the convective core if the difference between the same invariants at the two sides of the interface was less than 0.00005. If the computed envelope did not fit on the convective core, a new C was then chosen by interpolation and the whole procedure of integration was repeated. It is obvious from the procedure described here that the radiation pressure in the convective core is neglected so that the ratio γ of the specific heats has its normal value of 5/3 and consequently the convective core can be represented by the polytropic gas sphere of index 1.5.

The results of the integration are tabulated in the left side of Table 1 for different boundary values β_0 (that is, β at $x = 1$). The corresponding values obtained by Kushwaha are also listed in the table. It is apparent from this table that the structure of the model changes very little with the boundary conditions.

Table 1

The Structure of Stellar Models of Mass $M = 10\ M_\odot$

Characteristic Value	$B = 6.3096$, $F = 1122.04$				$B = 6.54$, $F = 1445.45$
	$\beta_0 = 0.1$	$\beta_0 = 0.5$	$\beta_0 = 0.9$	Kushwaha	Simultaneous Determination of L and R ($\beta_0 = 0.5$)
x_f	0.2322	0.2326	0.2329	0.2324	0.2386
$10^{-2}\ p_f$	0.2759	0.2741	0.2723	0.2747	0.2565
q_f	0.2444	0.2444	0.2443	0.2444	0.2533
t_f	0.5732	0.5722	0.5713	0.5725	0.5615
v_f	0.7165	0.7164	0.7163	0.7164	0.7370
w_f	1.7228	1.7227	1.7225	1.7226	1.7608
$10^7\ C$	2.63474	2.63399	2.63323	2.63494	2.20427
$\log \frac{L}{L_\odot}$				3.4769	3.3917
$\log \frac{R}{R_\odot}$				0.5594	0.5593
$\log T_c$				7.4416	7.4376
$\log \rho_c$				0.8919	0.8766

The purpose of the present detailed examination of the effect of boundary conditions arises from the possible existence of extensive corpuscular radiation, or simply outflow of gas, from B-type stars (Reference 8). The presence of these surface phenomena obviously modify the value of β_0 at the photospheric surface. Thus, the question of whether they affect the interior structure of the star does arise. From the results of the present calculation it appears that surface phenomena do not alter in any appreciable degree the interior structure – although they may have overwhelming influence on the course of its evolution, through the loss of mass (Reference 9). An obvious consequence of corpuscular radiation or gas streaming should, however, be noted. The energy generated in the interior by thermonuclear reactions is no longer equal to the luminosity but should be equal to the total rate of energy dissipation through outgoing matter as well as through radiation. Consequently, the mass of a star is greater than what would be expected from its luminosity alone.

SIMULTANEOUS DETERMINATION OF THE LUMINOSITY AND RADIUS

As is well known, both the luminosity and the radius of the star should be fixed once its mass and chemical composition are assigned. In the present section we propose to determine exactly from the boundary conditions these two quantities for a stellar model of 10 solar masses having the chemical composition given by Equations 15, that is, the same values as used by Kushwaha. This gives

$$B = B_0 = 6.54 \; . \tag{18}$$

Since, as has been shown in the previous section, the boundary value β_0 does not significantly affect the structure of the interior, the value of β_0 was set at $\beta_0 = 0.5$ for starting integration in all of the following calculations.

The luminosity of the star is given by Equation 8 in terms of the eigenvalue C which is determined by the boundary conditions of the star. On the other hand, the star, being in a steady state, must generate the same amount of energy by thermonuclear reactions as is radiated away. For the stars of 10 solar masses, in which the carbon cycle dominates the energy production that may be regarded as occurring solely in the convective core, the total energy generated is given by Kushwaha (Reference 2) as

$$L = \frac{1}{4\pi}\left(\frac{HG}{k}\right)^{16} \epsilon_0 \, \frac{\mu^{16} M^{18}}{R^{19}} \, K_f \int_0^{\xi_1} \xi^2 \theta^{19} \, d\xi \; , \tag{19}$$

where $\epsilon_0 = \epsilon_0' XZ$ with

$$\epsilon_0' = \frac{8 \times 10^{-144}}{3} \; , \tag{20}$$

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ξ_1 is the Emden radius of a purely convective star (e.g., Reference 6); and K_f is defined by

$$K_f = \left(\frac{x}{\xi}\right)^3 \beta^2 p^2 \frac{t^{14}}{\theta^{19}} , \tag{21}$$

all quantities on the right side of Equation 21 being evaluated at the interface.

In the derivation of Equation 19 the radiation pressure in the convective core has been set equal to that at the interface. Here the internal inconsistency of the Kushwaha treatment, which has been followed in this paper, is encountered. While we set $\beta = 1$ in the core in order to use the configuration of a polytropic gas sphere of index 1.5, we take $\beta = \beta_f$ in the calculation of energy generation in the same core. However, the actual value of β_f as seen in Table 1 is indeed near unity. This makes the inconsistency less serious than it would be otherwise. A correct way would be, of course, to take the radiation pressure in the convective core into consideration and to treat the fitting procedure accordingly (References 10, 11, and 12). It is apparent from what Savedoff and van Dyck (Reference 4) have reported that they have performed the calculation which takes into account the radiation pressure in the core; but the details of their calculations have not yet been published thus far.

The integral in Equation 19 can easily be evaluated from the configuration of a polytropic gas sphere of index 1.5, and is found to be 0.0778. It is now apparent that Equations 8 and 19, solved simultaneously, determine both the radius and the luminosity of the star once its mass and chemical composition are specified.

Equations 8 and 19 cannot be solved easily, because they involve constants such as C and K_f which can be determined by integration only after R, or equivalently F, has been given. This is the reason R was assumed in the previous investigations. However, with the use of high-speed digital computers, such a difficulty can be easily surmounted; thus, we propose the following procedure to carry out the simultaneous determination of L and R. First eliminate R from Equations 8 and 19 with the aid of Equation 9 and treat L and F instead of L and R as the two parameters to be determined, because it is F, not R, that enters explicity in Equations 4 through 6. Next, integrate these equations and determine four pairs of the eigenvalue C and K_f by the fitting procedure described in the preceding section for four different values of F chosen near the value assumed by Kushwaha. The determined values of K_f and C, together with the corresponding values of F, are given in Table 2. Once K_f and C are known, two values of $\log(L/L_\odot)$, which are also given in Table 2, can be derived from Equations 8 and 19. In other words, two relations between F and $\log(L/L_\odot)$ are obtained in the tabulated form, from which two interpolation formulas can be obtained as follows:

$$\log \frac{L}{L_\odot} = 3.08329 + 0.39594\,(10^{-3}\,F) - 0.16485\,(10^{-3}F)^2 + 0.02667\,(10^{-3}\,F)^3, \qquad (22)$$

which corresponds to Equation 8, and

$$\log \frac{L}{L_\odot} = -26.37492 + 33.68095(10^{-3}\,F) - 11.62845(10^{-3}\,F)^2 + 1.78083(10^{-3}\,F)^3, \qquad (23)$$

which corresponds to Equation 19. In solving these two simultaneous equations we obtain the luminosity and the radius of the star to a high degree of approximation. Thus,

$$\left.\begin{aligned} \log \frac{L}{L_\odot} &= 3.3917 \\ \text{and} \quad & \\ F = F_0 &= 1.44545 \times 10^3 \,. \end{aligned}\right\} \qquad (24)$$

Using this correct value of F we perform the integration once more, and the results are given in the last column of Table 1. It is apparent that $\log(R/R_\odot)$ as determined here is very near to what Kushwaha has assumed (cf. Table 1).

Table 2

The Values of C and K_f Corresponding to Different Values of F

10^{-3} F	K_f	10^7 C	$\log \frac{L}{L_\odot}$ from Equation 8	$\log \frac{L}{L_\odot}$ from Equation 19
1.3	0.842689	2.37477	3.378005	1.670736
1.4	0.803863	2.25480	3.387676	2.873266
1.5	0.769356	2.14664	3.396290	3.992817
1.6	0.738403	2.04855	3.404007	5.040074

Using Kushwaha's procedure, Morton (Reference 9) has studied the effect on the luminosity and radius of the energy-generation coefficient ϵ_0'. He studied two possibilities, (a) the case of ϵ_0' ten times that given by Equations 20 and (b) the case of ϵ_0' one tenth that given by Equations 20, for three cases of chemical composition:

$$X = 0.80, \quad Z = 0.02;$$

$$X = 0.70, \quad Z = 0.03;$$

$$X = 0.71, \quad Z = 0.02.$$

The differences in luminosity and in radius of the two possibilities (a) and (b) are equal to

$$\left.\begin{aligned} \Delta \log R &= 0.108 , \\ \Delta \log L &= -0.054 , \end{aligned}\right\} \qquad (25)$$

for all three cases. In other words, these differences are not sensitive to the change in chemical composition. Following the present procedure of simultaneous determination of L and R, we may investigate either of these possibilities simply by adding 1 to, or subtracting 1 from, the right side of Equation 23 and determining the corresponding luminosity and radius by solving the resulting equation simultaneously with Equation 22. The differences in luminosity and radius of the two possibilities (a) and (b) thus derived are:

$$\left.\begin{aligned} \Delta \log R &= 0.1079 , \\ \Delta \log L &= -0.0155 . \end{aligned}\right\} \qquad (26)$$

In spite of a hundredfold increase in the energy-generation coefficient the change in luminosity is small because $\log (L/L_\odot)$ as given by Equation 22 is a slowly varying function of F.

PERTURBATION CALCULATIONS

Following the method developed in a previous paper (Reference 13) for calculating the perturbed eigenvalue as a result of small changes in some physical parameters that enter into the problem of stellar interior, we may similarly study the perturbation in the present case by first linearizing Equations 4 through 9. However, for the practical purposes these perturbations can be derived in a much easier manner than that procedure. Instead of finding δC (the infinitesimal variation in C, resulting from the infinitesimal variations δB and δF) by integrating the linearized equations under some appropriate boundary conditions as was suggested in the previous paper, we may derive the perturbations in an approximate way. The eigenvalue $C = C_0 + \Delta C$ may first be derived by integrating the original Equations 4 through 6 under the same boundary conditions as given by Equations 11 and 13 but with $B = B_0 + \Delta B$ and $F = F_0 + \Delta F$. Here the symbol Δ denotes, as usual, the small but finite change; and C_0 is the eigenvalue given in the last column of Table 1, corresponding to the unperturbed configuration defined by B_0 and F_0 which are given respectively by Equations 18 and 24. Then the infinitesimal variation δC can be obtained in terms of δB and δF simply by interpolation, provided that the values of $C = C_0 + \Delta C$ have been calculated for a series of values of ΔB and of ΔF, respectively. The results of this are listed in Table 3 from which the following equations are obtained:

$$\frac{\delta K_f}{K_f} = a_1 \frac{\delta B}{B} + a_2 \frac{\delta F}{F} \tag{27}$$

and

$$\frac{\delta C}{C} = b_1 \frac{\delta B}{B} + b_2 \frac{\delta F}{F} , \tag{28}$$

with

$$a_1 = -0.4438, \quad a_2 = -0.6355 , \tag{29}$$

$$b_1 = -0.1006, \quad b_2 = -0.7121 , \tag{30}$$

at $B = B_0$, and $F = F_0$.

Table 3
The Values of C and K_f
Corresponding to Different Values of B and F

$F - F_0 = 1445.45$			$B - B_0 = 6.54$		
B	K_f	10^7 C	F	K_f	10^7 C
6.30	0.800552	2.21241	1425.45	0.794646	2.22621
6.42	0.794078	2.20834	1435.45	0.791118	2.21518
6.54	0.787634	2.20427	1445.45	0.787634	2.20427
6.66	0.781255	2.20021	1455.45	0.784191	2.19346
6.78	0.774943	2.19618	1465.45	0.780783	2.18277

Since all the physical parameters, such as the mass of the star, its chemical composition, the opacity coefficients κ_1', κ_2' and the energy-generation coefficient ϵ_0', enter into the equations that govern the stellar structure through the two parameters B and F, it is possible to deduce the effect of these physical parameters on the luminosity, the radius, and consequently the effective temperature, all from Equations 27 through 30 together with other variational equations which will be given below separately in different cases.

The Mass-Luminosity and Mass-Radius Relations Near $M = 10\ M_\odot$

Keeping the chemical composition, opacity coefficients, and energy-generation coefficients constant, we vary the mass of the star and investigate the effect on its luminosity and radius. It follows from Equations 7, 9, and 19 that

$$\frac{\delta B}{B} = 2\frac{\delta M}{M} , \tag{31}$$

$$\frac{\delta L}{L} = 5.5\frac{\delta M}{M} - 0.5\frac{\delta R}{R} + \frac{\delta C}{C} , \tag{32}$$

$$\frac{\delta F}{F} = 2.5\frac{\delta M}{M} - 0.5\frac{\delta R}{R} , \tag{33}$$

$$\frac{\delta L}{L} = 18\frac{\delta M}{M} - 19\frac{\delta R}{R} + \frac{\delta K_f}{K_f} . \tag{34}$$

The following results are obtained by eliminating $\delta B/B$, $\delta F/F$, $\delta K_f/K_f$ and $\delta C/C$ from the six Equations 27, 28 and 31 through 34:

$$(37 - b_2 + a_2)\frac{\delta R}{R} = \left[25 - 5(b_2 - a_2) - 4(b_1 - a_1)\right]\frac{\delta M}{M} \tag{35}$$

and

$$\begin{aligned}(37 - b_2 - a_2)\frac{\delta L}{L} = \Big[&191 - 3(b_2 - a_2) + 2(b_1 - a_1) + 74\, b_1 \\ &+ 80\, b_2 + 2\, b_1 a_2 - 2\, b_2 a_1\Big]\frac{\delta M}{M} .\end{aligned} \tag{36}$$

Equations 35 and 36 reduce to

$$\left.\begin{aligned}\frac{\delta R}{R} &= 0.6476\frac{\delta M}{M} , \\ \frac{\delta L}{L} &= 3.4253\frac{\delta M}{M}\end{aligned}\right\} \tag{37}$$

after the substitution of the values a_1, a_2, b_1, and b_2 given by Equations 29 and 30. Equations 37 lead directly to the variation in effective temperature

$$\left.\begin{aligned}\frac{\delta T}{T} &= 0.5325\frac{\delta M}{M} , \\ &\text{or} \\ \frac{\delta L}{L} &= 6.432\frac{\delta T}{T} .\end{aligned}\right\} \tag{38}$$

The latter defines the slope of the main sequence on the $\log L$ vs $\log T$ diagram. The results given by Equations 37 may be compared with the following statistical results derived from empirical data for the main sequence stars of a wide range of stellar masses by Russell and Moore (Reference 14):

$$\log L = 3.816 \log M - 0.244 \tag{39}$$

and

$$\log R = 0.700 \log M - 0.022 \tag{40}$$

in solar units.

The Effect of the Chemical Composition

The effect of chemical composition on the luminosity and radius of a star of constant mass may be similarly derived by taking variations of Equations 7, 8, 9 and 19. The resulting equations are:

$$\frac{\delta B}{B} = 4 \frac{\delta\mu}{\mu} , \tag{41}$$

$$\frac{\delta L}{L} = 7.5 \frac{\delta\mu}{\mu} - \frac{\delta Z}{Z} - \frac{\delta X}{1+X} - 0.5 \frac{\delta R}{R} + \frac{\delta C}{C} , \tag{42}$$

$$\frac{\delta F}{F} = -0.5 \frac{\delta R}{R} - \frac{\delta Z}{Z} + 3.5 \frac{\delta\mu}{\mu} , \tag{43}$$

$$\frac{\delta L}{L} = \frac{\delta X}{X} + \frac{\delta Z}{Z} + 16 \frac{\delta\mu}{\mu} - 19 \frac{\delta R}{R} + \frac{\delta K_f}{K_f} . \tag{44}$$

From the definition of μ, an auxiliary equation is obtained:

$$\frac{\delta\mu}{\mu} = \frac{-5\delta X + \delta Z}{3 + 5X - Z} . \tag{45}$$

The preceding five equations together with Equations 27 and 28 determine the variations in luminosity and radius in terms of those in X and Z. Thus, the following equations are obtained:

$$(37 - b_2 + a_2) \frac{\delta R}{R} = \left(2 + \frac{2X}{1+X} - 5\alpha X\right) \frac{\delta X}{X} + \left[4 + 2(b_2 - a_2) + \alpha Z\right] \frac{\delta Z}{Z} \tag{46}$$

and

$$(37 - b_2 + a_2)\frac{\delta L}{L} = \left[-(b_2 + 1) - \frac{(a_2 + 38)X}{1 + X} + 5\beta X\right]\frac{\delta X}{X}$$

$$+ \left[-39(b_2 + 1) - \beta Z\right]\frac{\delta Z}{Z} \quad , \tag{47}$$

where

$$\alpha = \frac{17 - 7(b_2 - a_2) - 8(b_1 - a_1)}{3 + 5X - Z}$$

and

$$\beta = \frac{(4a_1 - 117)(b_2 - a_2 - 37) - (a_2 + 38)(4b_1 - 4a_1 + 121)}{3 + 5X - Z} \quad .$$

These equations yield the numerical results

$$\frac{\delta R}{R} = -0.1602\frac{\delta X}{X} + 0.1043\frac{\delta Z}{Z} \tag{48}$$

and

$$\frac{\delta L}{L} = -3.2175\frac{\delta X}{X} - 0.2968\frac{\delta Z}{Z} \tag{49}$$

when the values of a_1 , a_2 , b_1, and b_2 given by Equations 29 and 30 have been substituted. Consequently,

$$\frac{\delta T}{T} = -0.7243\frac{\delta X}{X} - 0.1264\frac{\delta Z}{Z} \quad . \tag{50}$$

The effect of X on the luminosity and radius of the stellar model comes mainly through the mean molecular weight, while the effect of Z comes mainly through the opacity. This is the underlying reason why the luminosity decreases with increase both in X and Z .

The Effect of the Opacity Coefficients

The equations determining the variations $\delta L/L$ and $\delta R/R$ as the result of small changes in the opacity coefficients, $\delta\kappa_1'/\kappa_1'$ and $\delta\kappa_2'/\kappa_2'$, may be similarly derived:

$$\frac{\delta B}{B} = 0 , \tag{51}$$

$$\frac{\delta L}{L} = -\frac{\delta \kappa_1'}{\kappa_1'} - 0.5\,\frac{\delta R}{R} + \frac{\delta C}{C} , \tag{52}$$

$$\frac{\delta F}{F} = \frac{\delta \kappa_2'}{\kappa_2'} - \frac{\delta \kappa_1'}{\kappa_1} - 0.5\,\frac{\delta R}{R} , \tag{53}$$

$$\frac{\delta L}{L} = -19\,\frac{\delta R}{R} + \frac{\delta K_f}{K_f} . \tag{54}$$

It follows from these equations that

$$(37 - b_2 + a_2)\,\frac{\delta R}{R} = 2(1 + b_2 - a_2)\,\frac{\delta \kappa_1'}{\kappa_1'} - 2(b_2 - a_2)\,\frac{\delta \kappa_2'}{\kappa_2'} \tag{55}$$

and

$$(37 - b_2 + a_2)\,\frac{\delta L}{L} = -38(1 + b_2)\,\frac{\delta \kappa_1'}{\kappa_1'} + (38\, b_2 - a_2)\,\frac{\delta \kappa_2'}{\kappa_2'} , \tag{56}$$

or

$$\frac{\delta R}{R} = 0.0498\,\frac{\delta \kappa_1'}{\kappa_1'} + 0.0041\,\frac{\delta \kappa_2'}{\kappa_2'} , \tag{57}$$

and

$$\frac{\delta L}{L} = -0.2951\,\frac{\delta \kappa_1'}{\kappa_1'} - 0.7127\,\frac{\delta \kappa_2'}{\kappa_2'} . \tag{58}$$

Consequently,

$$\frac{\delta T}{T} = -0.0987\,\frac{\delta \kappa_1'}{\kappa_1'} - 0.1802\,\frac{\delta \kappa_2'}{\kappa_2'} . \tag{59}$$

Thus, an increase in the opacity coefficients diminishes the luminosity of the star but slightly enlarges its size. As a result, the effective temperature drops.

It should be noted here that the opacity coefficient, owing to electron scattering, is well known and is not expected to change its value. The term $\delta\kappa_2'/\kappa_2'$ is included in the present consideration for the reason that the actual opacity in the envelope may be better represented by a κ_2' different from its true value, just as Kushwaha has found in the present case that a factor $d = 1.21$ should be introduced in Equations 3 in order to have a better fit to the overall opacity.

The Effect of the Energy-Generation Coefficient

The equations of variation for the effect of the energy-generation coefficient are:

$$\frac{\delta B}{B} = 0 \;, \tag{60}$$

$$\frac{\delta L}{L} = -0.5\,\frac{\delta R}{R} + \frac{\delta C}{C} \;, \tag{61}$$

$$\frac{\delta F}{F} = -0.5\,\frac{\delta R}{R} \;, \tag{62}$$

$$\frac{\delta L}{L} = \frac{\delta \epsilon_0'}{\epsilon_0'} - 19\,\frac{\delta R}{R} + \frac{\delta K_f}{K_f} \;, \tag{63}$$

which give

$$(37 - b_2 + a_2)\,\frac{\delta R}{R} = 2\,\frac{\delta \epsilon_0'}{\epsilon_0'} \tag{64}$$

and

$$(37 - b_2 + a_2)\,\frac{\delta L}{L} = -2(1 + b_2)\,\frac{\delta \epsilon_0'}{\epsilon_0'} \;, \tag{65}$$

or

$$\left.\begin{aligned} \frac{\delta R}{R} &= 0.0539\,\frac{\delta \epsilon_0'}{\epsilon_0'} \;, \\ \frac{\delta L}{L} &= -0.0078\,\frac{\delta \epsilon_0'}{\epsilon_0'} \;, \end{aligned}\right\} \tag{66}$$

and

$$\frac{\delta T}{T} = -0.0289\,\frac{\delta \epsilon_0'}{\epsilon_0'} \;. \tag{67}$$

Thus, should the energy-generation coefficient increase by a small amount, the luminosity would decrease slightly and the radius increase slightly and consequently the effective temperature drop a little. This conclusion can be seen also from the results of direct calculation given by Equations 26. The differences $\Delta \log R$ and $\Delta \log L$ between the two possibilities considered by Morton and mentioned previously may be roughly estimated from Equations 66 by setting $\Delta \log \epsilon_0' = 1$ and $\Delta \log \epsilon_0' = -1$ and taking the difference. Thereby we obtain

$$\left.\begin{aligned} \Delta \log R &= 0.1078 \,, \\ \Delta \log L &= 0.0156 \,, \end{aligned}\right\} \qquad (68)$$

which agree surprisingly well with Equation 26 although a tenfold increase or decrease can hardly be regarded as a small variation for which Equations 66 actually apply. In any case, these results clearly indicate that the effect of the energy-generation coefficient on the luminosity of the model is insignificant.

ILLUSTRATION OF THE VARIOUS EFFECTS ON THE H-R DIAGRAM

The relative importance of the percentage changes in various physical parameters $\delta M/M$, $\delta X/X$, $\delta Z/Z$, $\delta\kappa_1'/\kappa_1'$, $\delta\kappa_2'/\kappa_2'$ and $\delta\epsilon_0'/\epsilon_0'$ in determining the location of the star on the Hertzsprung-Russell diagram is shown in Figure 1. The point P is the location of the unperturbed model and was determined earlier in the discussion on simultaneous determination of the luminosity and radius. The various arrows starting from this point indicate the direction and relative magnitude with which the star's location on the diagram would change in response to the indicated change of the various parameters.

Since the percentage changes are very different for the different parameters, the effect of $\delta Z/Z$ and $\delta\kappa_1'/\kappa_1'$ has been magnified ten times and the effect of $\delta\epsilon_0'/\epsilon_0'$ has been magnified one hundred times in the diagram. The figure reveals the fact that, percentage-wise, the mass and the hydrogen content are the two major factors in determining the location of the star on the H-R diagram. Thus, except for the very small change arising from the variation of the energy-generation coefficient, changes arising from other parameters follow quite closely the slope defined by the main sequence. This fact certainly helps the stars of the zero-age main sequence to fall on a narrow line, even though they may have different chemical compositions, and consequently different opacity coefficients.

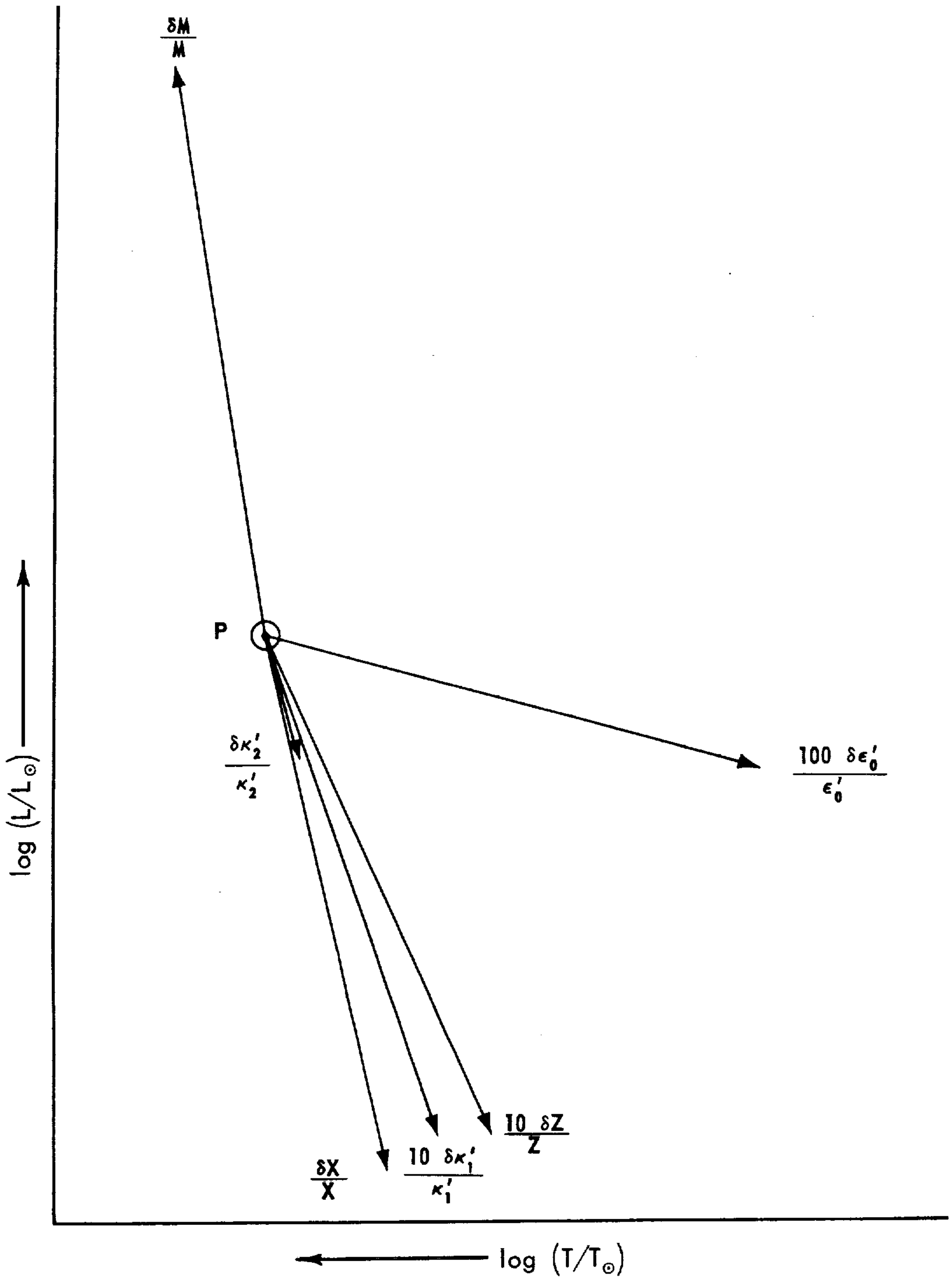

Figure 1—Displacements of a star on the H-R diagram as a result of the changes in its various parameters. The point P represents the location of the star, while different arrows starting from this point indicate the direction as well as the relative magnitudes of the star's displacement on the diagram in response to the indicated percentage changes of the various parameters. In the figure the effect of $\delta Z/Z$ and $\delta\kappa_1'/\kappa_1'$ has been magnified ten times and the effect of $\delta\epsilon_0'/\epsilon_0'$ has been magnified one hundred times.

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